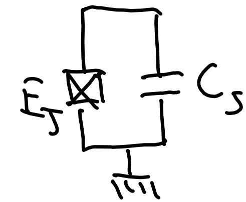
From CPB to TRANSMON

By replacing the geometric inductance L of an LC oscillator by a Josephson junction it makes the circuit nonlinear.



In this situation, the energy levels of the circuit are no longer equidistant.

If the non linearity and the quality factor of the junction are large enough, the energy spectrum resembles that of an atom, with well-resolved and nonuniformly spread spectral lines. We therefore often refer to this circuit as a "superconducting artificial atom". In many situations we can restrict our attention to only two energy levels, typical the ground and the first excited state, forming a qubit.

To make this discussion more precise, it is useful to see how the Hamiltonian of the circuit is modified by the presence of the Josephson junction.

While the energy stored in a linear inductor is $E = \int dt\, V(t) I(t) = \int dt \left(d\Phi/dt \right) I = \Phi^2/2L$, where we have used $\Phi = LI$, the energy of the non linear inductance is:

$$E = I_c \int dt \left(\frac{d\Phi}{dt} \right) \sin\left(\frac{2\pi}{\Phi_0} \Phi \right) = -E_J \cos\left(\frac{2\pi}{\Phi_0} \Phi \right),$$

with $E_J = \Phi_0 I_c / 2\pi$ the Josephson energy. This quantity is proportional to the rate of tunneling of Cooper pairs across the junction.

Taking into account this contribution, the quantized Hamiltonian of the capacitively shunted Josephson junction reads: $\hat{H}_T = \frac{(\hat{Q} - Q_g)}{2C_\Sigma} - E_J \cos\left(\frac{2\pi}{\Phi_0} \hat{\Phi} \right) = 4E_c (\hat{n} - n_g)^2 - E_J \cos\hat{\varphi}$

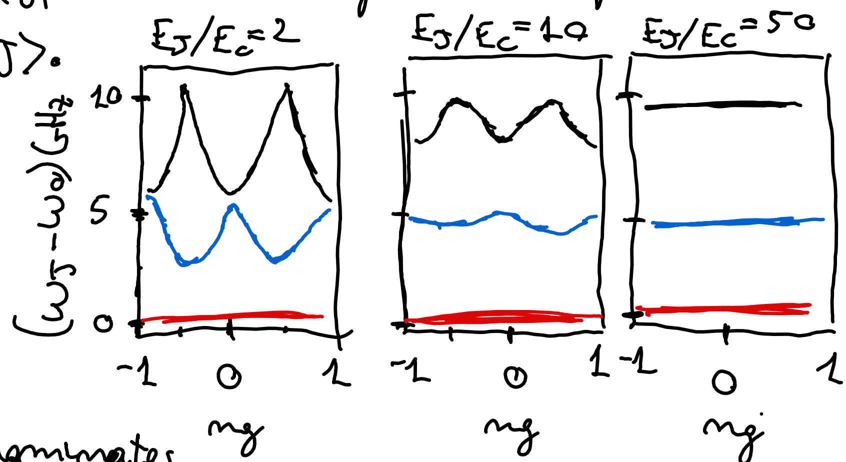
In this expression $C_\Sigma = C_J + C_S$ is the total capacitance including the junction's capacitance C_J and the shunt capacitance C_S. We have defined the "charge number" operator $\hat{n} = \hat{Q}/2e$, the "phase operator" $\hat{\varphi} = (2\pi/\Phi_0)\hat{\Phi}$ [mod 2π], and the charging energy $E_C = e^2/2C_\Sigma$. We have also included a possible offset charge $n_g = Q_g/2e$ due to capacitive coupling of the transmon to external charges.

The offset charge can arise from spurious unwanted degrees of freedom in the transmon's environment or from an external gate voltage $V_g = Q_g/C_g$.

The choice of E_J and E_C is crucial in determining the system's sensitivity to the offset charge.

The spectrum of $\hat{H}_T$ is controlled by the ratio E_J/E_C, with different values of this ratio corresponding to different types of superconducting qubits. Regardless of the parameter regime, one can always express the Hamiltonian in the diagonal form $\hat{H} = \sum_J \hbar\omega_J |J\rangle\langle J|$ in terms of its eigenfrequencies ω_J and eigenstates $|J\rangle$.

In Fig. we plot the energy difference $(\omega_J - \omega_0)$ for the 3 lowest energy levels for different ratios E_J/E_C.



If the charging energy dominates, $E_J/E_C < 1$, the eigenstates of the Hamiltonian are approximately given by eigenstates of the charge operator, $|J\rangle \simeq |n\rangle$, with $\hat{n}|n\rangle = n|n\rangle$.

In this situation, a change in gate charge n_g has a large impact on the transition frequency of the device.
As a result, unavoidable charge fluctuations in the circuit's environment lead to corresponding fluctuations in the qubit transition frequency, and consequently to dephasing.
To mitigate this problem, a solution is to work in the "transmon regime", where the ratio E_J/E_C is large.
Typical values in the experiments are $E_J/E_C \sim 20-80$.
In this situation, the charge degree of freedom is highly delocalized due to the large Josephson energy. For this reason, the first energy levels of the device become essentially independent of the gate charge.
It can be in fact shown that the charge dispersion, which describes the variation of the energy levels with gate charge, decreases exponentially with E_J/E_C in the transmon regime.

The net results is that the coherence time of the device is much larger than at small E_J/E_C.

However, the price to pay for this increased coherence is the reduced anharmonicity of the transmon.

A high enough anharmonicity is necessary to control the qubit without causing unwanted transitions to higher excited states.

Fortunately, while charge dispersion is exponentially small with E_J/E_C, the loss of anharmonicity has a much weaker dependence on this ratio, given by $\sim (E_J/E_C)^{-1/2}$.

Because of the gain in coherence, the reduction in anharmonicity is not an impediment to controlling the transmon state with high fidelity.

While the variance of the charge degree of freedom is large when $E_J/E_C \gg 1$, the variance of its conjugate variable $\hat{\varphi}$ is correspondingly small with $\Delta\hat{\varphi} = \sqrt{\langle\varphi^2\rangle - \langle\varphi\rangle^2} \ll 1$.

Given this condition, it is instructive to rewrite

$$\hat{H}_T = \underbrace{4E_C\hat{n}^2 + \frac{1}{2}E_J\hat{\varphi}^2}_{\text{LC harmonic oscillator}} - E_J\left(\cos\hat{\varphi} + \frac{1}{2}\hat{\varphi}^2\right)$$

The first two terms correspond to an LC circuit of capacitance C_Σ and inductance $E_J^{-1}(\Phi_0/2\pi)^2$, the linear part of the Josephson inductance.

We have dropped the offset charge n_g on the basis that the frequency of the relevant low-lying energy levels is insensitive to this parameter. It is still possible to use an externally oscillating voltage source to cause transition between the transmon states.

The last term in $\hat{H}_T$ is the non linear correction to this harmonic potential which, for $E_J/E_C \gg 1$ and therefore $\Delta\hat{\varphi} \ll 1$ can be truncated to its non linear correction $\hat{H}_q = 4E_C\hat{n}^2 + \frac{1}{2}E_J\hat{\varphi}^2 - \frac{1}{4!}E_J\hat{\varphi}^4$.

As expected, the transmon can be represented as a weakly anharmonic oscillator.
Note that in this approximation, the phase $\hat{\varphi}$ is sufficiently localized, which holds for low-lying energy eigenstates in the transmon regime with $E_J/E_C \gg 1$.
Given this approximation, it is then useful to introduce creation and annihilation operators chosen to diagonalize the first in $\hat{H}_q$: $\hat{\varphi} = \left(\frac{2E_C}{E_J}\right)^{1/4}(\hat{b}^\dagger + \hat{b})$; $\hat{n} = \frac{i}{2}\left(\frac{E_J}{2E_C}\right)^{1/4}(\hat{b}^\dagger - \hat{b})$
This form makes it quite clear that fluctuations of the phase $\hat{\varphi}$ decrease with E_J/E_C, while the reverse is true for the conjugate charge operator $\hat{n}$.

Using this expressions, we will get:

$$\hat{H}_q = \sqrt{8E_CE_J}\,\hat{b}^+\hat{b} - \frac{E_C}{12}(\hat{b}^+ + \hat{b})^4 \approx \hbar\omega_q \hat{b}^+\hat{b} - \frac{E_C}{2}\hat{b}^+\hat{b}^+\hat{b}\hat{b}$$

where $\hbar\omega_q = \sqrt{8E_CE_J} - E_C$.

We kept only terms that have the same number of creation and annihilation operators. This is reasonable because, in a frame rotating at ω_q, any terms with an unequal number of $\hat{b}$ and $\hat{b}^+$ will be oscillating. If the freq. of these oscillations is large, then these terms rapidly averages out. This is known as Rotating Waves Approximation (RWA)

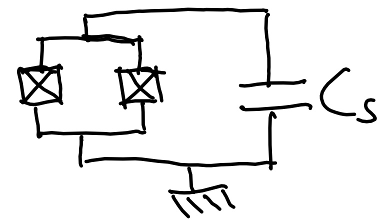
The quantity $\omega_p = \sqrt{8E_CE_J}/\hbar$ is known as the Josephson plasma frequency and corresponds to the frequency of small oscillations of the "effective particle mass" C at the bottom of a well of the cosine potential of the Josephson junction. In the transmon regime, this frequency ω_p get renormalized by the charging energy: $\omega_q = \omega_p - E_C/\hbar$

The term $-\frac{E_C}{2}\hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b}$ is a Kerr non linearity.
with $E_C/\hbar$ playing the role of Kerr frequency shift.
per excitation of the nonlinear oscillator.
The anharmonicity of the transmon is $-E_C$, with typical values $E_C/\hbar \sim 100-400$ MHz. While this non linearity is small with respect to the oscillator frequency ω_q, it is in practice much larger than the spectral linewidth that can routinely be obtained for these artificial atoms and can therefore easily be spectrally resolved. As a resul, and in contrast to more traditional realizations of Kerr nonlinearities in quantum optics, it is possible with superconducting quantum circuits to have a large Kerr nonlinearity, even at the single-photon level.

For quantum information processing, the presence of this nonlinearity is necessary to address only the ground and first excited state, without unwanted transition to other states.
In this case, the transmon acts as a two-level system, or a qubit.
However, it is important to keep in mind that the transmon is a multilevel system and that it is often necessary to include higher levels in the description of the device to quantitatively explain experimental observations.

Flux tunable TRANSMON

A useful variant of the transmon artificial atom is the flux-tunable transmon, where the single Josephson junction is replaced with two parallel junctions forming a superconducting quantum interference device (SQUID).



The transmon Hamiltonian then reads

$$\hat{H}_T = 4E_C\,\hat{n}^2 - E_{J1}\cos\hat{\varphi}_1 - E_{J2}\cos\hat{\varphi}_2$$

where E_{Ji} is the Josephson energy of the i-junction and $\hat{\varphi}_i$ the phase difference across that junction.

In the presence of an external flux Φ_x threading the SQUID loop, flux quantization requires

$$\hat{\varphi}_1 - \hat{\varphi}_2 = 2\pi\,\Phi_x/\Phi_0 \pmod{2\pi}.$$

Defining the average difference $\hat{\varphi} = (\hat{\varphi}_1 + \hat{\varphi}_2)/2$

we have: $$\hat{H}_T = 4E_C \hat{n}^2 - E_J(\Phi_x)\cos(\hat{\varphi} - \varphi_e)$$

where $$E_J(\Phi_x) = E_{J\Sigma}\cos\left(\frac{\pi\Phi_x}{\Phi_0}\right)\sqrt{1 + d^2\tan^2\left(\frac{\pi\Phi_x}{\Phi_0}\right)},$$

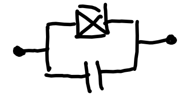
with $E_{J\Sigma} = E_{J2} + E_{J1}$, and $d = (E_{J2} - E_{J1})/E_{J\Sigma}$ represents the junctions asymmetry.

Therefore, by replacing a single junction with a SQUID loop yields an effective flux-tunable Josephson energy $E_J(\Phi_x)$. In turn, this results in a flux tunable transmon frequency

$$\omega_q(\Phi_x) = \left[\sqrt{8E_C|E_J(\Phi_x)|} - E_C\right]/\hbar$$

The transmon frequency can be tuned by as much as ~GHz in as little as 10-20 ns. This possibility will be explored for bringing qubit frequency into resonance to implement qubits logic gates.

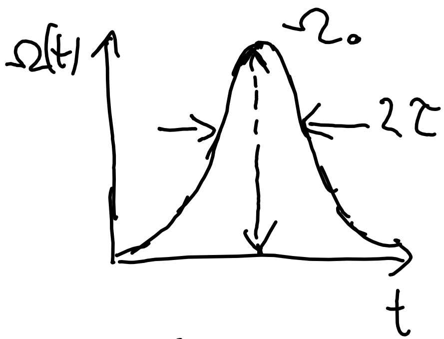
Using an anharmonic oscillator as a qubit

We have found that the electrical circuit  is described by an effective Hamiltonian that in the limit $E_C \ll E_J$ (called "transmon-limit") is well approximated by an anharmonic oscillator

$$H \simeq \hbar \omega_{01} a^\dagger a - \hbar \frac{\alpha}{2} a^{\dagger 2} a^2$$

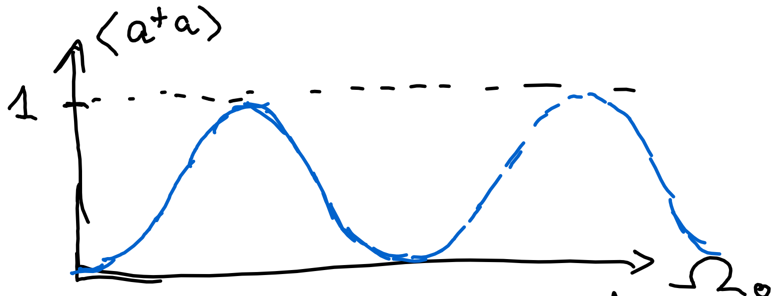
In order to see that such a system can be operated as an effective qubit, we study the dynamics under the influence of a <u>drive field</u> applied resonantly with the transmon transition frequency $\omega_{drive} = \omega_{01}$.
Such a drive field is typically realized as a time varying voltage $V(t) = V_0(t) \cos(\omega_0 t + \varphi)$ applied across the transmon. The corresponding driven Hamiltonian is $\hat{H}_{drive}(t) = \hat{Q}\, V(t) \simeq \hbar \Omega(t) \left(a\, e^{i(\omega_{01} t + \varphi)} + a^\dagger e^{-i(\omega_{01} t + \varphi)} \right)$

Let's assume that $\Omega(t)$ follows Gaussian envelope

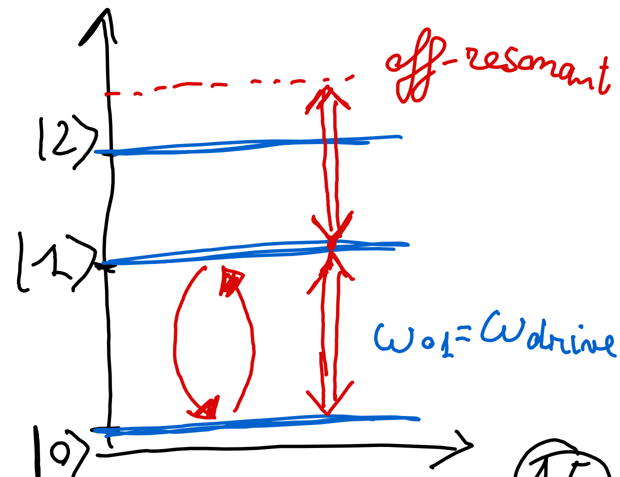


with $2\tau \gg \frac{1}{\lambda}$ (small bandwidth)

Evolving an initial ground state $|0\rangle$ ($\simeq |g\rangle$) according to this Hamiltonian results in so called "Rabi oscillations" between $|0\rangle$ and $|1\rangle$ ($\simeq |e\rangle$).



Due to finite λ, higher excited states do not get populated.



In contrast, for a linear system with $\lambda = 0$, an initial state $|0\rangle$ evolves into a (classical) coherent state $|\beta\rangle$ with amplitude $\beta \sim \Omega_0$.

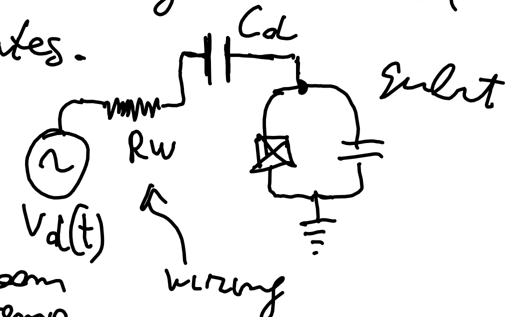
In the presence of a large λ, we can describe the system as an effective 2-level system

$$H = \hbar \omega_{01} \frac{\hat{\sigma}_z}{2}$$

By choosing the drive amplitude Ω_0, time and phase φ appropriately, we can perform any rotation of the Bloch vector about any axis in the xy-plane.

Single-qubit gate:

Here we will review the steps necessary to demonstrate that capacitive coupling of MW to a superconducting circuit can be used to drive single-qubit gates.



Capacitive coupling: X, Y control

$$H = -\frac{\omega_q}{2}\sigma_z + \Omega V_d(t)\,\sigma_y = H_0 + H_{int}$$

$$\Omega = (C_d/C_\Sigma)\,Q_{zpf} \qquad \omega_q = (E_1 - E_0)/\hbar$$

To elucidate the role of the driving, we move into a frame rotating with the qubit at freq. ω_q ("rotating frame" or "interaction frame"). To see the usefulness of this frame, consider a state $|\psi_0\rangle = (1\ \ 1)^T/\sqrt{2}$ (on the equatorial frame). Considering the propagator U_{H_0} corresponding to H_0, by the time dependent Schröd. eq. we have

$$|\psi_0(t)\rangle = U_{H_0}|\psi_0\rangle = 1/\sqrt{2}\begin{pmatrix} e^{i\omega_q t/2} \\ e^{-i\omega_q t/2}\end{pmatrix}.$$

By calculating e.g. $\langle \psi_0 | \sigma_x | \psi_0 \rangle = \cos(\omega_q t)$, it is evident that the phase is winding with a frequency ω_q due to the σ_z term. By going into a frame rotating with the qubit at frequency ω_q, the action of the driving can be more clearly studied.

We can do this by $U_{rf} = e^{i H_0 t} = U_{H_0}^{\dagger}$. The new state in the rotating frame is $|\psi_{rf}(t)\rangle = U_{rf} |\psi_0\rangle$

and the Schröd. eq. is now

$$
\begin{aligned}
i \partial_t |\psi_{rf}(t)\rangle &= i (\partial_t U_{rf}) |\psi_0\rangle + i U_{rf} (\partial_t |\psi_0\rangle), = \\
&= i \dot{U}_{rf} U_{rf}^{\dagger} |\psi_{rf}\rangle + U_{rf} H_0 |\psi_0\rangle = \\
&= \underbrace{\left(i \dot{U}_{rf} U_{rf}^{\dagger} + U_{rf} H_0 U_{rf}^{\dagger} \right)}_{\tilde{H}_0} |\psi_{rf}\rangle
\end{aligned}
$$

$\tilde{H}_0$ is the transformed H_0 in the new rotating frame.

We can see that the new $\tilde{H}_0 = 0$, as expected because the rotating frame takes care of the time dependence.

This transformation allows also to obtain the new form of the interaction term in rotating frame

$$\tilde{H}_d = \Omega V_d(t) [\cos(\omega_q t)\sigma_y - \sin(\omega_q t)\sigma_x]$$

We can assume that the time dependent part of the voltage, $V_d(t) = V_0 v(t)$, has the generic form

$$v(t) = S(t) \sin(\omega_d t + \phi) =$$
$$= S(t) [\cos(\phi)\sin(\omega_d t) + \sin(\phi)\cos(\omega_d t)]$$

dimentionless envelope function that sets the shape and amplitude of the drive $\underline{V_0 S(t)}$

$I = \cos(\phi)$ ["in-phase" component]

$Q = \sin(\phi)$ ["out-phase" component]

We can rewrite the driving Hamiltonian as

$$\tilde{H}_d = \Omega V_0 S(t)\left[I \sin(\omega_d t) - Q\cos(\omega_d t)\right] \times$$
$$\times \left[\cos(\omega_q t)\sigma_y - \sin(\omega_q t)\sigma_x\right].$$

RWA: we can dropp the fast rotating terms $(\omega_q + \omega_d)$ that will average to zero.

$$\tilde{H}_d = \frac{1}{2}\Omega V_0 s(t)\Big[\big(-I\cos(\delta\omega t) + Q\sin(\delta\omega t)\big)\sigma_x +$$
$$\delta\omega = \omega_q - \omega_d \qquad + \big(I\sin(\delta\omega t) - Q\cos(\delta\omega t)\big)\sigma_y\Big]$$

$$\tilde{H}_d = -\frac{\Omega}{2} V_0 S(t)\begin{bmatrix} 0 & e^{i(\delta\omega t+\phi)} \\ e^{-i(\delta\omega t+\phi)} & 0 \end{bmatrix}$$

This last expression for the driving Hamiltonian in the rotating frame is a powerful tool for understanding single qubit gates in superconducting qubits.

- Let's apply a pulse at the qubit frequency

$\delta\omega = 0$: $\tilde{H}_d = -\frac{\Omega}{2} V_0 S(t) (I \sigma_x + Q \sigma_y)$

$I \to \cos(\phi)$, $Q \to \sin(\phi)$

This shows that an in-phase pulse ($\phi = 0$, or the I-quadrature) gives a rotation around the X-axis; while an out-of-phase pulse ($\phi = \frac{\pi}{2}$, or the Q-quadrature) corresponds to a rotation around the y-axis.

- As a concrete example of an in-phase pulse, we can define the unitary operation

$$U_d^{\phi=0}(t) = \exp\left[\left(\frac{i}{2} \Omega V_0 \int_0^t S(t')\,dt'\right) \sigma_x\right]$$

which depends on the envelope of the pulse $S(t)$ and amplitude V_0 that are controlled using the AWG. This $U_d^{\phi=0}$ is known as "Rabi" driving.

(AWG = Arbitrary Waveform Generator)

We can define $\Theta(t) = -\Omega V_0 \int_0^t S(t')dt'$ representing the angle by which a state is rotated given the circuit parameters, the magnitude V_0 and the shape of the pulse envelope $S(t)$.

This means that to implement a π-pulse around the x-axis, one would solve the eq. $\Theta(t) = \pi$ and output the signal in-phase on the qubit driving line.

In this framework, a sequence of pulses $\Theta_k, \Theta_{k-1}, \dots, \Theta_0$ is converted to a sequence of gates operating on a qubit

$$U_k \cdots U_1 U_0 = \prod_{n=0}^{k} \exp\left[-\frac{i}{2}\Theta_n(t)\left(I_n \sigma_x + Q_n \sigma_y\right)\right]$$